

Bikei Invariants and Gauss Diagrams for Virtual Knotted Surfaces

Sam Nelson*

Patricia Rivera†

Abstract

Marked vertex diagrams provide a combinatorial way to represent knotted surfaces in \mathbb{R}^4 ; including virtual crossings allows for a theory of virtual knotted surfaces and virtual cobordisms. Biquandle counting invariants are defined only for marked vertex diagrams representing knotted orientable surfaces; we extend these invariants to all virtual marked vertex diagrams by considering colorings by involutory biquandles, also known as bikei. We introduce a way of representing marked vertex diagrams with Gauss diagrams and use these to characterize orientability.

KEYWORDS: marked vertex diagrams, ch-diagrams, knotted surfaces, virtual knotted surfaces, bikei

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1 Introduction

Biquandles are algebraic structures with axioms motivated by the oriented Reidemeister moves. In particular, the set of labelings of the semiarcs in a oriented knot diagram by elements of a finite biquandle satisfying a certain labeling condition forms a computable invariant known as the *biquandle counting invariant*. Biquandles were introduced in [4] and have been much studied in recent years; see [3, 10] etc. for more.

Marked vertex diagrams, also known as *ch-diagrams*, are planar diagrams similar to ordinary knot diagrams which encode a knotted surface in \mathbb{R}^4 [11–13]. It has been recently established ([7]) that two marked vertex diagrams represent ambient isotopic surfaces in \mathbb{R}^4 if and only if they are related by a sequence of the *Yoshikawa moves* introduced in [13]. Marked vertex diagrams can be used to represent both closed knotted surfaces and cobordisms between knots and have advantages over some other popular methods of representing knotted surfaces in \mathbb{R}^4 : they are easier to draw than broken surface diagrams and require only a single diagram, unlike movie diagrams which require multiple diagrams. See [5, 11–13] for more.

Virtual knots, also known as *abstract knots*, are a combinatorial generalization of knots including *virtual crossings* representing genus in the ambient space of the knot. Including virtual crossings in marked vertex diagrams yields *virtual knotted surfaces*; see [6, 8, 9] for more.

Bikei, also known as *involutory biquandles*, were introduced in [1] as a special case of biquandles which can be used to define invariants of unoriented knots and links. Unlike the more general case of biquandles, bikei counting invariants are defined for all virtual marked vertex diagrams, regardless of topological type. In this paper, we study bikei counting invariants for virtual knotted surfaces.

Gauss diagrams (and equivalently *Gauss codes*) are a computer-friendly way of representing virtual knot diagrams. We define Gauss diagrams for marked vertex diagrams and use these to characterize orientability.

The paper is organized as follows. In Section 2 we review the basics of bikei. In Section 3 we review knotted surfaces and marked diagrams. In Section 4 we define bikei counting invariants for marked vertex diagrams. In Section 5 we introduce Gauss diagrams for marked vertex diagrams and use them to characterize orientability. We end in Section 6 with some questions for future research.

*Email: knots@esotericka.org. Partially supported by Simons Foundation collaboration grant 316709

†Email: patriciariv25@gmail.com

2 Bikei

We begin with a standard definition.

Definition 1. Let X be a set. A *biquandle structure* on X is a pair of binary operations $(x, y) \mapsto x^y, x_y$ such that for all $x, y, z \in X$ we have

- (i) $x^x = x_x$,
- (ii) the maps $\alpha_y, \beta_y : X \rightarrow X$ and $S : X \times X \rightarrow X \times X$ defined by $\alpha_y(x) = x_y$, $\beta_y(x) = x^y$ and $S(x, y) = (y_x, x^y)$ are bijective, and
- (iii) the *exchange laws*

$$\begin{aligned} (x^y)^{(z^y)} &= (x^z)^{(y_z)} \\ (x^y)_{(z^y)} &= (x_z)_{(y_z)} \\ (x_y)_{(z_y)} &= (x_z)_{(y^z)} \end{aligned}$$

A biquandle such that $x_y = x$ for all x, y is a *quandle*. A biquandle such that $(x^y)^y = x = (x_y)_y$, $x^{yy_x} = x^y$ and $y_{x^y} = y_x$ for all $x, y \in X$ is a *bikei*.

Remark 1. We are using the original notation for biquandles from [4]; much recent work uses similar-looking but different notation resulting in less symmetric axioms.

We will be primarily interested in the case of bikei.

Example 1. Let $n \in \mathbb{Z}$. Any group G is a bikei with $x^y = yx^{-1}y$ and $x_y = x$ as well as with $x_y = x$ and $x^y = yx^{-1}y$.

Example 2. Any abelian group A with elements s, t satisfying $s^2 = t^2 = 1$ and $1 + t = s(1 + t)$ is a bikei with

$$x^y = tx + (s - t)y, \quad x_y = sx$$

Such a bikei is called an *Alexander bikei*.

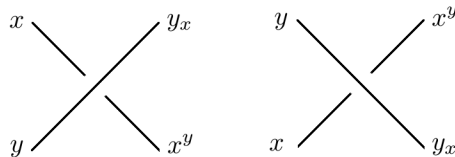
Bikei structures on a set $X = \{x_1, \dots, x_n\}$ can be specified using an $n \times 2n$ matrix encoding the operation tables of the biquandle operations. More precisely, let $M_{i,j} = k$ where

$$x_k = \begin{cases} (x_i)^{x_j} & 1 \leq j \leq n \\ (x_i)_{x_j} & n+1 \leq j \leq 2n \end{cases}$$

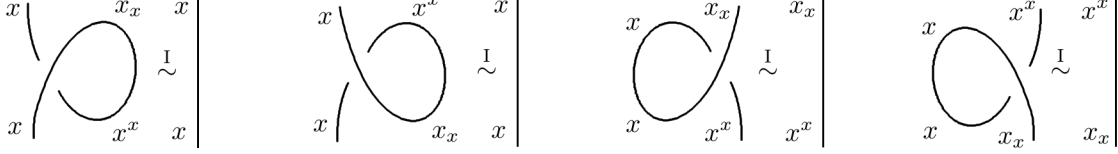
Example 3. The Alexander bikei $X = \mathbb{Z}_4 = \{x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 0\}$ with $s = 3$ and $t = 1$ has biquandle matrix

$$\left[\begin{array}{cccc|cccc} 3 & 1 & 3 & 1 & 3 & 3 & 3 & 3 \\ 4 & 2 & 4 & 2 & 2 & 2 & 2 & 2 \\ 1 & 3 & 1 & 3 & 1 & 1 & 1 & 1 \\ 2 & 4 & 2 & 4 & 4 & 4 & 4 & 4 \end{array} \right]$$

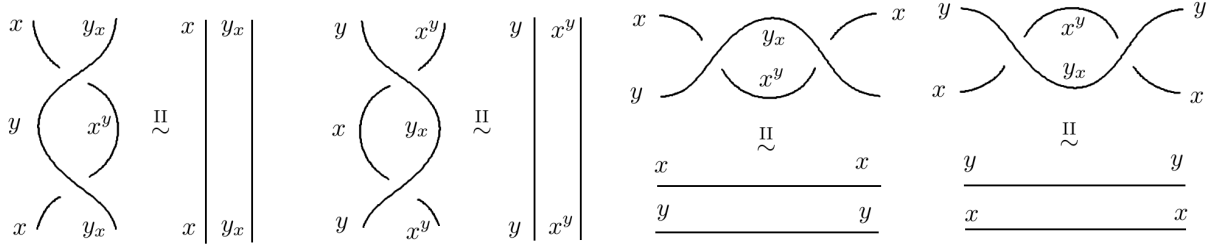
The bikei axioms come from the unoriented Reidemeister moves. We think of elements of X as labels for the semiarcs in an unoriented knot or link diagram with operations as depicted.



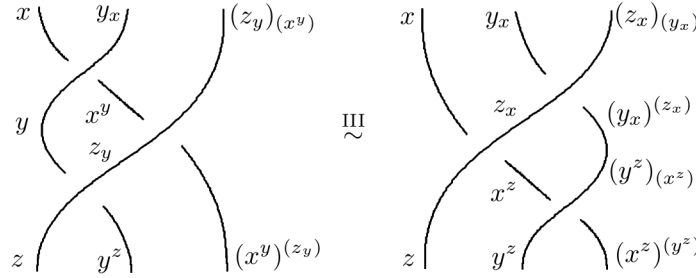
Then the Reidemeister I move requires that for all $x \in X$, we have $x^x = x_x$:



The conditions $(x^y)^y = (x_y)_y = x$, $x^y = x^{y_x}$ and $x_y = x_{y^x}$ satisfy the Reidemeister II moves.



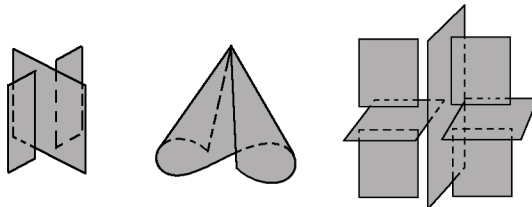
The Reidemeister III move implies the exchange laws:



For oriented knots and links, biquandle colorings by a finite biquandle define a computable invariant known as the *biquandle counting invariant*. For unoriented knots and links L , the number of bikei colorings of a diagram of L by a finite bikei X is an invariant known as the *bikei counting invariant*, denoted $\Phi_X^{\mathbb{Z}}(L)$. We will extend this invariant to virtual knotted surfaces in following sections.

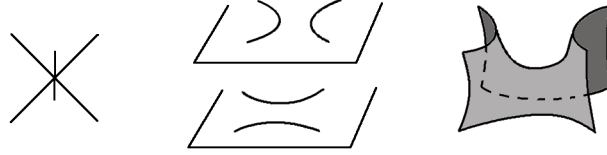
3 Knotted Surfaces

A *knotted surface* is a smoothly embedded compact surface $\Sigma \subset \mathbb{R}^4$ with finitely many components. Knotted surfaces can be represented with *broken surface diagrams* analogous to knot diagrams where a broken sheet indicates the sheet crosses under in the fourth dimension much as a broken strand in a knot diagram indicates the strand crossing under in the third dimension. The self-intersection set in the projection into \mathbb{R}^3 can include endpoints and triple points as well as closed curves. Alternatively, knotted surfaces can be represented with *movie diagrams* and *braid charts*; see [2] for much more.

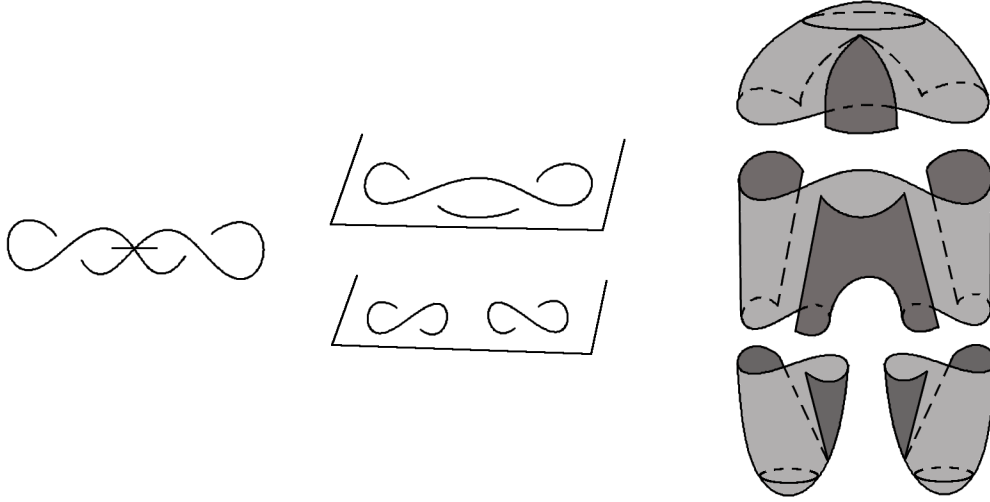


Every knotted surface can be moved by ambient isotopy into a position such that all of its maxima in the x_3 (say) direction are in the $x_3 = 1$ hyperplane, all of its minima are in the $x_3 = -1$ hyperplane and all of its saddle points are in the $x_3 = 0$ hyperplane; such a position is known as a *hyperbolic splitting*. In particular each cross-section of the surface by a hyperplane of the form $x_3 = \epsilon$ for $\epsilon \in (-1, 0) \cup (0, 1)$ is an unlink, with the (x_1, x_2) cross-sections forming Reidemeister move sequences ending with crossingless unlink diagrams near $\epsilon = \pm 1$.

We can represent a knotted surface in such a position with a *marked vertex diagram* or *ch-diagram*, a diagram consisting of ordinary crossings together with *saddle crossings* representing saddle points:

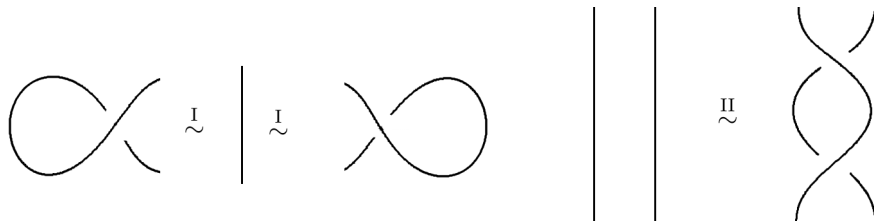


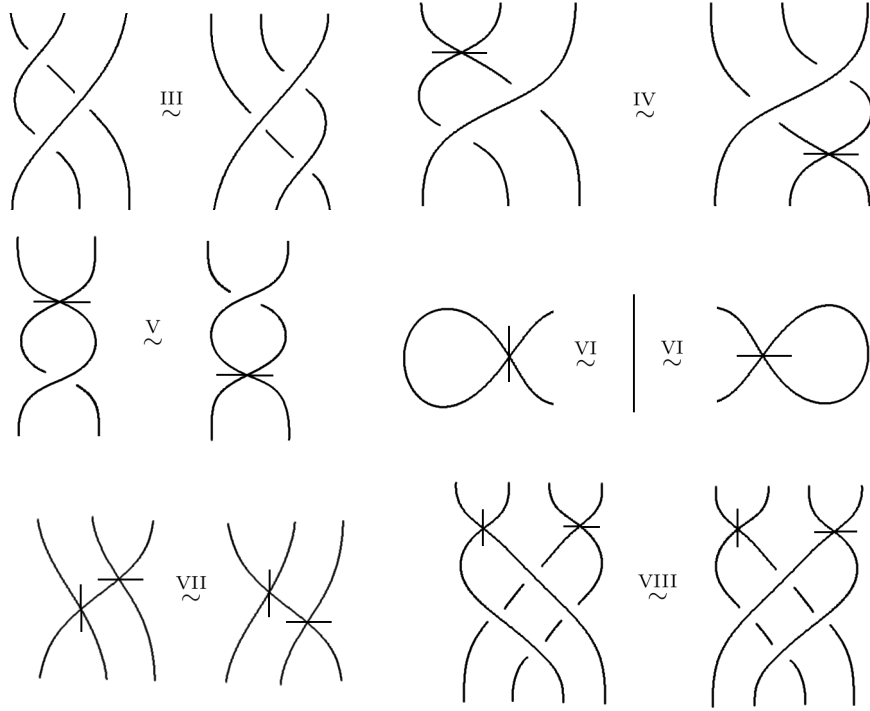
To recover a broken surface diagram from a marked vertex diagram, we resolve the saddle crossings into saddles with the crossings resolving into crossed sheets; near the $x_3 = 0$ hyperplane, we have a cobordism between unlinks. These unlinks then resolve over the intervals $x_3 \in (-1, 0) \cup (0, 1)$ into disjoint circles which we cap off with maxima and minima. See [11–13] for more.



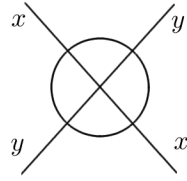
Remark 2. Marked vertex diagrams which yield nontrivial knots or links L, L' after smoothing the saddle crossings do not correspond to knotted closed surfaces but to *corbordisms* between L and L' , i.e., surfaces whose boundaries are $L \cup L'$.

In [13], Yoshikawa introduced a set of moves now known as *Yoshikawa moves* and conjectured that two knotted surfaces are ambient isotopic iff their marked vertex diagrams are related by the Yoshikawa moves. This conjecture has recently been established [7]. There are eight moves, the first three of which are the usual Reidemeister moves:

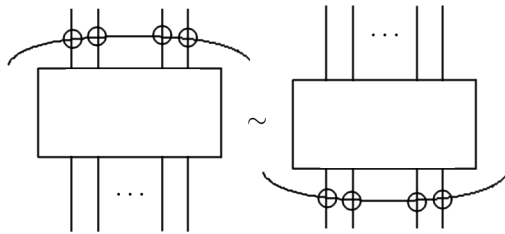




In [13], marked vertex diagrams are required to yield unlink diagrams after smoothing the saddles in order to obtain closed knotted surfaces; as we have observed, relaxing this requirement yields cobordisms between the links obtained after smoothing. In [8] this idea is used to extend the notion of cobordism to the case of *virtual knots*, knots and links with *virtual crossings*, which represent genus in the ambient space rather than points where the knot is close to itself within the ambient space. We will not break our semiarcs at virtual crossings, so the biquandle labeling rule is as depicted.



Virtual crossings interact with classical and saddle crossings via the *detour move*



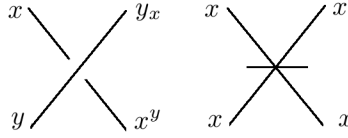
which says that any portion of the knot with only virtual crossings can be replaced with any other strand with the same endpoints and only virtual crossings. In particular, marked vertex diagrams with virtual crossings represent cobordisms between the virtual links obtained by smoothing the saddle crossings.

Finally, we note that there is an analog of crossing number for virtual marked vertex diagrams; in [13], the minimal number of classical crossings for a given surface-knot type K is denoted $c(K)$, and the minimal number of saddle crossings is denoted $h(K)$ (for “hyperbolic points”); then the *ch-index* is $ch(K) = c(K) + h(K)$.

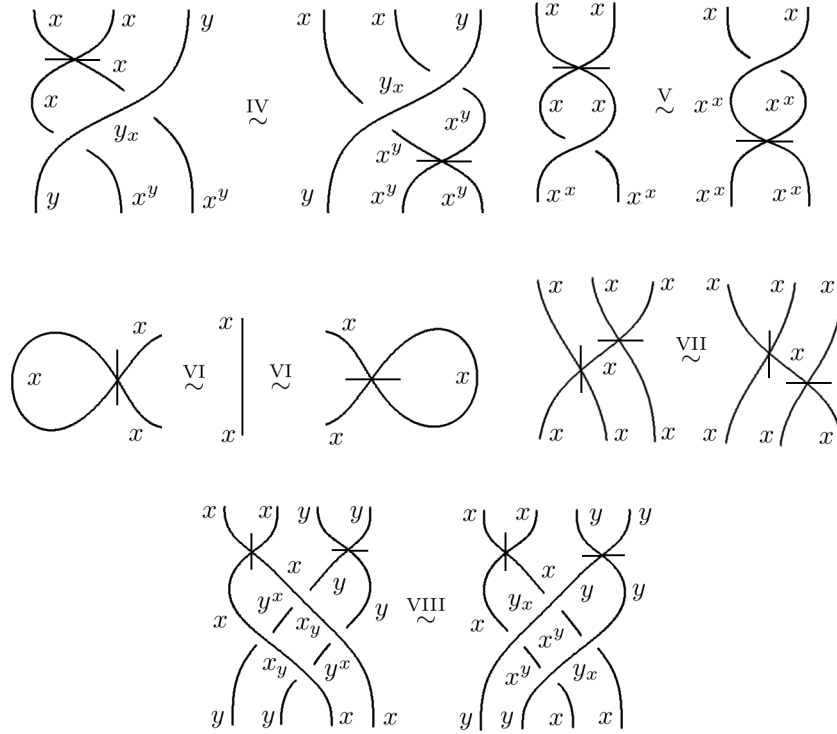
Definition 2. The *virtual ch-index* of a virtual knotted surface K , $vch(K)$, is the minimal total number of virtual, classical, and saddle crossings over all diagrams D of K .

4 Bikei Counting Invariants

Let X be a bikei and L a marked vertex diagram. An X -labeling of L is an assignment of elements of X to the semiarcs of L (the portions of the diagram between the crossing points) satisfying the rules pictured below:



Checking the remaining Yoshikawa moves, we have

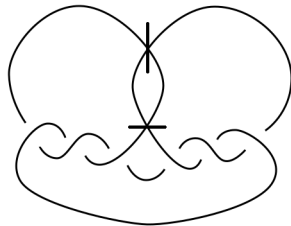


Moreover, X -labelings are unaffected by the detour move. Thus we obtain:

Theorem 1. If X is a bikei and L and L' are marked vertex diagrams related by Yoshikawa moves then for every X -labeling of L there is a unique corresponding X -labeling of L' .

Corollary 2. Let X be a finite bikei. The number of X -labelings of a virtual marked vertex diagram is an invariant of virtual knotted surfaces.

Example 4. The diagram 8_1 below represents the *spun trefoil*, the surface swept out in \mathbb{R}^4 by spinning a trefoil knot about an axis.

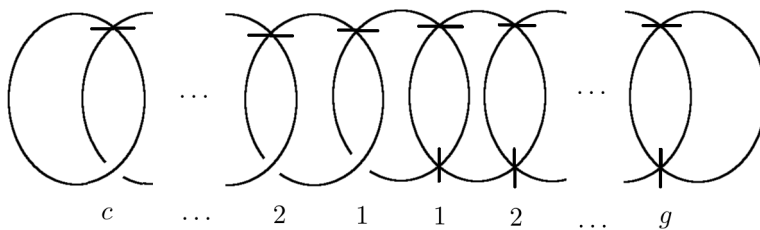


It is distinguished from the unknotted sphere 0_1 by the bikei counting invariant with bikei X given by the matrix

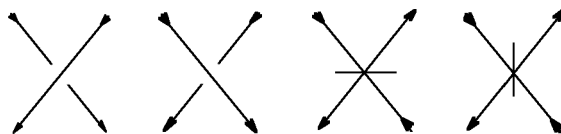
$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 2 & 2 & 3 & 2 & 4 & 3 \\ 2 & 3 & 3 & 3 & 2 & 4 & 3 & 2 \\ 4 & 4 & 4 & 4 & 4 & 3 & 2 & 4 \end{array} \right]$$

with $\Phi_X^{\mathbb{Z}}(8_1) = 10 \neq 4 = \Phi_x^{\mathbb{Z}}(0_1)$.

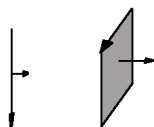
One important difference between surface knot theory and classical knot theory is that in classical knot theory, there is only one object to knot, namely the circle S^1 (or disjoint unions of copies of S^3), while in surface knot theory there are already infinitely many topologically distinct surfaces before knotting, characterized by cross-cap number c and genus g .



An orientation for a virtual marked vertex diagram is a choice of orientation for each semiarc such that at classical crossing we have a “pass-through” rule at at each saddle crossing we have a “source-sink rule”:



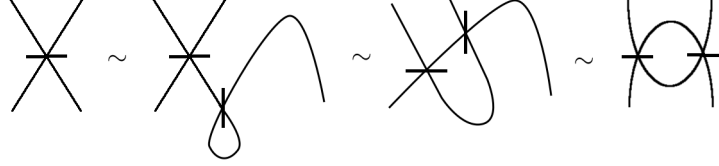
These define a local orientation on the surface by defining a choice of normal vector on each sheet.



If the surface is not orientable, these local orientations cannot be extended to a global orientation.

Definition 3. The closed curves obtained by “passing through” both saddle and classical crossings in a virtual marked vertex diagram are its *naïve components*.

Remark 3. We note that the number of naïve components is not invariant under Yoshikawa moves, since the following move splits or merges two naïve components:



See also [13].

Since an unknotted orientable surface S of genus g has a diagram with only saddle crossings, every coloring of S by any bikei X is monochromatic. Thus we have:

Proposition 3. *For any unknotted orientable surface S of genus $g \in \mathbb{N}$ and bikei X , the bikei counting invariant of S is $\Phi_X^{\mathbb{Z}} = |X|$.*

For non-orientable unknotted surfaces, we likewise need monochromatic colorings, but the classical crossings present an obstruction: a monochromatic coloring of such a surface by $x \in X$ is only valid if $x = x^x = x_x$. Then we have:

Proposition 4. *For any unknotted non-orientable surface S of genus $g \in \mathbb{N}$ and cross-cap number c and bikei X , the bikei counting invariant of S is $\Phi_X^{\mathbb{Z}} = |F|$ where*

$$F = \{x \in X \mid x^x = x_x = x\}.$$

Corollary 5. *If a bikei X is a kei, i.e. an involutory quandle, then $\Phi_X^{\mathbb{Z}} = |X|$ for all unknotted surfaces S .*

The set $F = \{x \in X \mid x^x = x_x = x\}$ is the union of singleton sub-bikeis of X . F can be all of X (as in the quandle case where $x^x = x$ for all $x \in X$), or it could be a proper sub-bikei; it can even be empty.

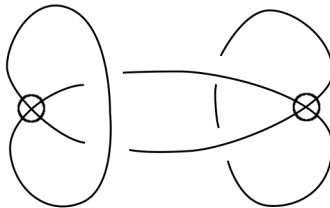
Example 5. Let $X = \mathbb{Z}_2$ and define $x^y = x_y = x + 1$. Then $\Phi_X^{\mathbb{Z}}(S) = 2$ for all unknotted orientable surfaces and $\Phi_X^{\mathbb{Z}}(S) = 0$ for all unknotted non-orientable surfaces. A coloring by X will be called a *2-coloring*.

Finally, we note that classical crossing changes do not affect 2-colorability. Thus, we have:

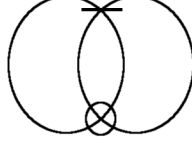
Proposition 6. *Knotted orientable surfaces which can be unknotted by classical crossing changes are 2-colorable; knotted non-orientable surfaces which can be unknotted by classical crossing changes are not 2-colorable.*

5 Gauss diagrams for virtual knotted surfaces

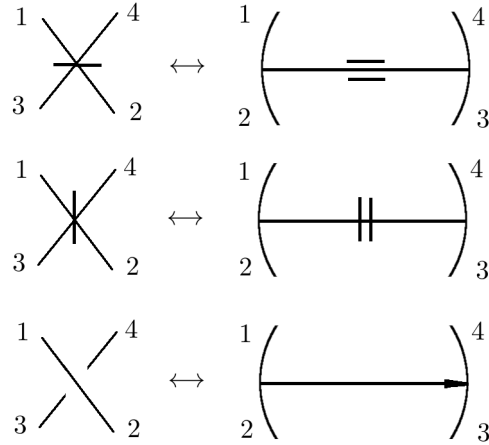
In light of proposition 6 it is natural to ask whether a virtual knotted surface is orientable if and only if it is 2-colorable. For virtual knots, it is known that classical crossing change is not an unknotting operation; for instance, the *Kishino virtual knot* below cannot be unknotted by classical crossing change.



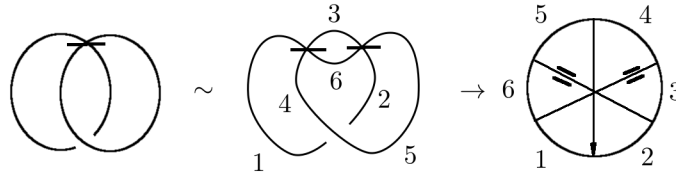
However, in the virtual knotted case orientability and 2-colorability need not coincide: the diagram below cannot be given a consistent source-saddle orientation, but is 2-colorable:



To characterize orientability for virtual knotted surface diagrams, we introduce Gauss diagrams for virtual marked vertex diagrams. In light of the move in remark 3, we can without loss of generality consider only diagrams with a single naïve component, which we write as a circle around which we write crossing labels representing over/under crossings points and two points representing each saddle point. The two instances of each crossing are then connected by chords, with the classical crossing chords directed toward the undercrossing point and the saddle crossing chords marked to indicate the direction of the saddle as depicted.



The portions of the circles between endpoints of either chords or arrows correspond to the semiarcs of the diagram.

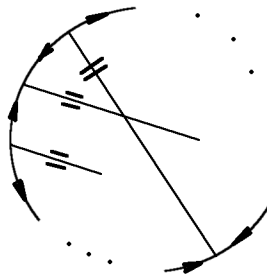


We then have the following characterization of orientability:

Proposition 7. *A marked vertex Gauss diagram is orientable iff for every chord C the number of chord endpoints between the two endpoints of C is even.*

Proof. Traveling around the naïve component of an oriented diagram, each chord endpoint reverses the local orientation. Since each chord represents a source-sink oriented saddle crossing, the local picture must be as

depicted:



Thus, we need an even number of switches between the ends of C on both sides.

Conversely, if every cord meets the stated criteria, then there are two choices of global orientation for the diagram, determined by choosing a direction for one semicircle and alternating at every chord endpoint; the condition that every chord has an even number of endpoints between its two ends implies that both such orientations are source-sink for each saddle crossing and pass-through for each classical crossing. \square

6 Questions

We conclude with some questions for future research.

- What kinds of enhancements of the bikei counting invariant are there? Cocycle enhancements, for instance, generally require orientation.
- What happens when we add nontrivial operations at the virtual crossings?

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DEPARTMENT OF MATHEMATICAL SCIENCES
 CLAREMONT MCKENNA COLLEGE
 850 COLUMBIA AVE.
 CLAREMONT, CA 91711